Intriguing Sets in Finite Classical Polar Spaces

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Outline I

Strongly regular graphs

Definition

A strongly regular graph with parameters (v, k, λ, μ) is a k-regular graph with v vertices such that two distinct vertices have exactly λ or μ common neighbors according as they are adjacent or not.

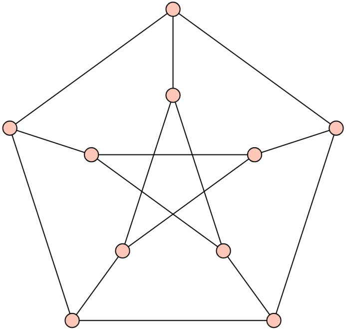


Figure: The Peterson Graph, $(v, k, \lambda, \mu) = (10, 3, 0, 1)$

Studies on strongly regular graphs[1]

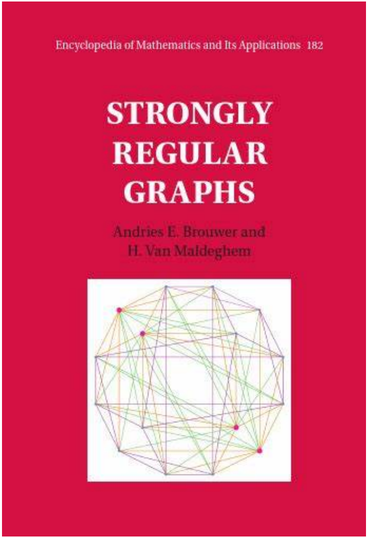
A strongly regular graph is

- a distance regular graph with diameter two;
- one of non-diagonal relations of a two-class symmetric association scheme.

In particular, any vertex-transitive graph with a rank-three permutation automorphism group is strongly regular.

[1] A. E. Brouwer and H. Van Maldeghem. Strongly Regular Graphs, Cambridge University Press, 2022. https://homepages.cwi.nl/ aeb/math/srg/rk3/

Studies on strongly regular graphs[2]



[2] A. E. Brouwer and H. Van Maldeghem. Strongly Regular Graphs, Cambridge University Press, 2022. https://homepages.cwi.nl/ aeb/math/srg/rk3/

Equitable partitions

Let Γ be a (primitive) SRG with parameters (v, k, λ, μ) with vertex set V and adjacency matrix A. The spectrum of Γ is $\{k^1, r^f, s^g\}$.

Definition

A partition $\{Y_1, Y_2\}$ of V is called equitable if each vertex of Y_i is adjacent to precisely e_{ij} vertices of Y_j, $1 \leq i, j \leq 2$.

For a subset Y of vertices in V, we define its characteristic row vector χ_Y such that $\chi_Y(v) = 1$ iff $v \in Y$. Then

$$\chi_{Y_1} A = e_{11}\chi_{Y_1} + e_{21}\chi_{Y_2}, \qquad \chi_{Y_2} A = e_{12}\chi_{Y_1} + e_{22}\chi_{Y_2}$$

or equivalently

$$(\chi_{Y_1} - \frac{|Y_1|}{|V|}\mathbf{1})A = (e_{11} - e_{12})(\chi_{Y_1} - \frac{|Y_1|}{|V|}\mathbf{1}).$$

Here $|Y_1|e_{12} = |Y_2|e_{21}$, which is obtained by double counting.

Regular sets

Definition

If Y is a set of vectices such that $Y, V \setminus Y$ form an equitable partition, then Y is a regular set of Γ.

Remark:

(1) The regular sets of Γ divide into two types according as the corresponding eigenvalue is r or s.

(2) If $H \leq \mathrm{Aut}(\Gamma)$ has two orbits on V, then the two orbits form an equitable partition and so each is a regular set.

Geometrically defined strongly regular graphs

Finite classical projective spaces

Let V be a vector space of dim n over $\mathbb{F} = \mathbb{F}_q$. The classical projective geometry $\mathrm{PG}(V)$ (or $\mathrm{PG}(n-1, q)$) consists of the following objects of $n-1$ different types:

1. points: the 1-dim subspaces of V;
2. lines: the 2-dim subspaces of V (if any);
3. planes: the 3-dim subspaces of V (if any);
4. solids: the 4-dim subspaces of V (if any);
5. $\cdots\cdots$;
6. hyperplanes: the $(n-1)$-dim subspaces of V (if any),

and the relation between the objects are defined via inclusion.

Theorem (Birkhoff-von Neumann)

Let κ be a non-degenerate σ-sesquilinear reflexive form on V. Up to a scalar factor, κ is one of the following types:

1. *Alternating:* $\kappa(u, u) = 0, \forall\, u \in V$.
2. *Symmetric:* $\kappa(u, v) = \kappa(v, u), \forall\, u, v \in V$.
3. *Hermitian:* $\kappa(u, v) = \kappa(v, u)^{\sigma}, \forall\, u, v \in V$, *where* $o(\sigma) = 2$.

Let κ be a non-degenerate form on $V = \mathbb{F}_q^n$ given as above .

1. A vector u is *isotropic* if $\kappa(u, u) = 0$.
2. A subspace U is *totally isotropic* if $\kappa(u, v) = 0$, $\forall u, v \in U$.
3. For a nonzero vector u, define its perp as

$$u^{\perp} = \{v \in V : \kappa(u, v) = 0\}.$$

The *polar space* associated to a nondegenerate form κ on V is the geometry whose *points, lines, planes,* $\cdots$ are the totally isotropic subspaces of V of *dimension* 1, 2, 3, $\cdots$.

Witt's Lemma

Theorem

Let $Q: V \to F$ be a quadratic form with a non-degenerate bilinear form. If $\theta : U_1 \to U_2$ is an isometry between two subspaces, then there is an isometry $\hat{\theta}: V \to V$, s.t., $\theta = \hat{\theta}|_{U_1}$.

As a corollary, the maximal t.s./t.i. subspaces of a polar space have the same (affine) dimension, called the rank of the polar space.

The symplectic polar space $W(2)$

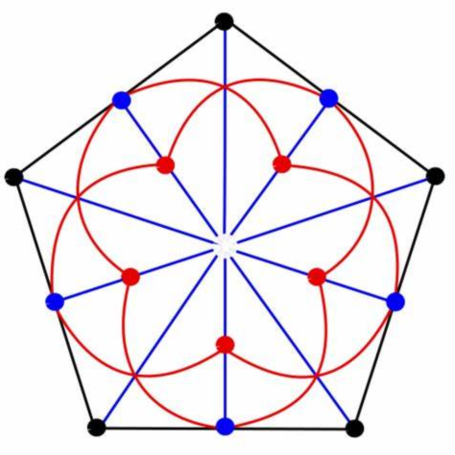


Figure: $W(2) = \mathrm{GQ}(2,2)$, $(v, k, \lambda, \mu) = (15, 6, 1, 3)$

The collinearity graph of polar spaces

Let Γ be the collinearity graph of the polar graph $\mathcal{P}$:

(a) vertex set: the isotropic or singular points;

(b) adjacency: $\langle x \rangle \sim \langle y \rangle$ iff $\kappa(x, y) = 0$.

By Witt's Lemma, it is routine to show that

Theorem

The graph Γ is a strongly regular graph.

Finite classical polar spaces

There are three types of finite classical polar spaces: orthogonal polar spaces, symplectic polar spaces and Hermitian polar spaces.

Table: Finite classical polar spaces

	$\mathcal{P}_r$	$n = \dim V$	ovoid number θ_r	ϵ
S	$W(2r-1,q)$	$2r$	q^r+1	0
O	$Q^+(2r-1,q)$	$2r$	$q^{r-1}+1$	1
	$Q^-(2r+1,q)$	$2r+2$	$q^{r+1}+1$	1
	$Q(2r,q)$	$2r+1$	q^r+1	1
U	$H(2r,q)$	$2r+1$	$q^{r+\frac{1}{2}}+1$	1/2
	$H(2r-1,q)$	$2r$	$q^{r-\frac{1}{2}}+1$	1/2

Each generator of $\mathcal{P}_r$ has size $\frac{q^r-1}{q-1}$ and $|\mathcal{P}_r| = \frac{q^r-1}{q-1}\theta_r$, where $\theta_r = q^{n-r-\epsilon}+1$

Intriguing sets (regular sets)

Suppose that $r \geq 2$, and let $\mathcal{M}$ be a nonempty set of $\mathcal{P}_r$. Then $\mathcal{M}$ is an *intriguing set* if there exist two constants h_1, h_2 such that

$$|P^{\perp} \cap \mathcal{M}| = \begin{cases} h_1, & P \in \mathcal{M} \\ h_2, & P \notin \mathcal{M}, \end{cases}$$

where P ranges over the points of $\mathcal{P}_r$ and $P^{\perp}$ is the set of points collinear with P.

J. Bamberg, S. Kelly, M. Law, and T. Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory Ser. A*, 114(7):1293–1314, 2007.

Intriguing sets

There are two types of intriguing sets:

(1) i-tight sets: $|\mathcal{M}| = i \cdot \frac{q^r-1}{q-1}$,

$$h_1 = q^{r-1} + i \cdot \frac{q^{r-1}-1}{q-1}, \quad h_2 = i \cdot \frac{q^{r-1}-1}{q-1};$$

(2) m-ovoids: $|\mathcal{M}| = m\theta_r$,

$$h_1 = (m-1)\theta_{r-1} + 1, \quad h_2 = m\theta_{r-1}.$$

J. Bamberg, S. Kelly, M. Law, and T. Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory Ser. A*, 114(7):1293–1314, 2007.

Example:

(1) $\mathcal{P}_r$ itselt is both a θ_r-tight set and $\frac{q^r-1}{q-1}$-ovoid (trivial example).

(2) Each generator (t.i. or t.s. r-space) is a 1-tight set.

(3) For two disjoint intriguing sets, their union is also an intriguing set.

Lemma

An i-tight set and an m-ovoid share mi points in common.

Sktech of proof: The eigenvectors with distinct eigenvalues are orthogonal.

J. Bamberg, S. Kelly, M. Law, and T. Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory Ser. A*, 114(7):1293–1314, 2007.

A short-cut method

Lemma (Feng, L., Xiang, 2024+)

Let $\mathcal{M}$ be a subset of size $i\frac{q^r-1}{q-1}$ or $m\theta_r$ in $\mathcal{P}_r$ for some integer i or m, and let h_1, h_2 be the corresponding parameters determined by $|\mathcal{M}|$. Then the following are equivalent:

(1) *$\mathcal{M}$ is an intriguing set in $\mathcal{P}_r$;*

(2) *$|P^\perp \cap \mathcal{M}| = h_1$ for all $P \in \mathcal{M}$;*

(3) *$|P^\perp \cap \mathcal{M}| = h_2$ for all $P \in \mathcal{P}_r \setminus \mathcal{M}$.*

Sketch of Proof: (1) $\Leftrightarrow$ (2), i.e., $\sum_{P\in\mathcal{P}_r\setminus\mathcal{M}} \left(|P^\perp \cap \mathcal{M}| - h_2\right)^2 = 0$.

T. Feng, L., and Q. Xiang. A new family of (q^4+1)-tight sets with an automorphism group $F_4(q)$, arXiv:2305:16119.

Outline I

Some geometrical constructions

Here we only list some examples of Hermitian polar space.

1. Non-degenerate hyperplane sections: Q is a nonsingular point.
   $Q^{\perp} \cap H(2r, q^2) \cong H(2r-1, q)$: $(q^{2r-1}+1)$-tight set;
   $Q^{\perp} \cap H(2r-1, q^2) \cong H(2r-2, q^2)$: $(q^{2r-1}-1)/(q^2-1)$-ovoids,
2. Subfield embeddings:
   $W(2r-1, q) \hookrightarrow H(2r-q, q^2)$: $(q+1)$-tight sets.
3. Field reduction:
   i-tight set in $H(2r-1, q^{2e}) \Rightarrow$ i-tight set in $H(2er-1, q^2)$
4. Derivation.

J. Bamberg, S. Kelly, M. Law, and T. Penttila. Tight sets and m-ovoids of finite polar spaces. *J. Combin. Theory Ser. A*, 114(7):1293–1314, 2007.

Transitive ovoid[3]

Theorem (Feng, L., 2021)

Suppose that O is a transitive ovoid of $H(n, q^2)$, $n \geq 3$ odd. Then $n = 3$, and O is projectively equivalent to one of the following:

(1) *a classical ovoid $H(2, q^2)$, with full stabilizer $\mathrm{P\Gamma U}(3, q^2)$;*

(2) *a Singer-type ovoid for even $q = 2^d$, with full stabilizer $C_{q^3+1} : 6d$ if $q > 2$;*

(3) *an exceptional ovoid of $H(3, 5^2)$, with full stabilizer $C_2 \times (C_3 \times \mathrm{PSL}(2, 7)) : 2$;*

(4) *an ovoid of $H(3, 8^2)$, with full stabilizer $C_{57} : 9$;*

(5) *an ovoid of $H(3, 8^2)$, with full stabilizer $C_{57} : 18$.*

Problem: Give the non-existence of ovoids of finite classical polar spaces of high rank (≥ 5)?

[3] T. Feng, L. On transitive ovoids of finite hermitian polar spaces. Combinatorica **41** (2021), 645-667.

Transitive m-ovoids with irreducible groups[4]

Let $H \leq \Gamma\mathrm{L}(V)$ be almost simple such that

$$L = H^{(\infty)},\ M = N_{\Gamma}(L).$$

$\mathcal{P}$	L	$\#L$-orbits	m	Condition
$Q(6,q)$	${}^2G_2(q)$	3	$1, q, q^2$	$p = 3,\ f$ odd
$Q(6,q)$	$\mathrm{SU}_3(q)$	2	$1, q + q^2$	p^3
$Q^+(7,q)$	$\mathrm{SL}_2(q^3)$	2	$1, q + q^2 + q^3$	$p = 2$
$Q^+(7,q)$	$\mathrm{SU}_3(q)$	5	$1,\ \{q + q^2\},\ q^3$	$q = 2 \pmod 3$
$W(3,q)$	$\mathrm{Sz}(q)$	2	$1, q$	$p = 2,\ f$ odd

The ovoids that appear in the table are the Ree ovoid, unitary ovoid, Desarguesian ovoid and Suzuki ovoid respectively.

[4] T. Feng, L., R. Tao, On m-ovoids of finite polar spaces with an irreducible transitive automorphism group, Sci. China Math., **67** (2024), no. 3, 683-712.

Cameron-Liebler line classes in PG$(3, q)$

The Klein correspondence is a bijection between lines of PG$(3, q)$ and singular points of $Q^+(5, q)$.

C-L line classes with parameter x

$$\Updownarrow$$

x-tight sets in $Q^+(5, q)$

Feng and Xiang et al developed a method to construct many interesting families of C-L lines classes (intriguing sets) with "small" automorphism groups.

T. Feng, K. Momihara, M. Rodgers and Q. Xiang, and H. Zou. Cameron-Liebler line classes with parameter $x = (q+1)^2/3$. *Adv. Math.* **85** (2021), Paper No. 107780, 31 pp.

The methods for constructing intriguing sets can be roughly classified into three types:

1. traditional geometrical methods.
2. group-theoretic methods: highly symmetry (CFSG).
3. the combination of geometrical and algebraic methods: “small” automorphism group.

Outline I

Our recent result

Theorem (Feng, L., Xiang, 2023+)

Let $q = p^f$ be a prime power. The classical polar space $Q(24, q)$ or $Q^-(25, q)$ admits a $(q^4 + 1)$-tight set with an automorphism group isomorphic to the exceptional group $F_4(q)$ according as $q = 3^f$ or $q = 2 \pmod 3$.

The novelty of our construction is the use of the exceptional group $F_4(q)$ on its minimal module over $\mathbb{F}_q$.

Octonions

The octonion $\mathcal{O}$ (Cayley number) is an 8-dimensional algebra with (orthonormal) basis $\mathbf{1}, e_1, \ldots, e_7$ satisfying $e_i^2 = -1$ and the multiplication derived by the following Fano plane.[5]
Let N be the norm of $\mathcal{O}$, $N(x) = x\bar{x}$. Then the form f is induced from N.
The automorphism group $\mathrm{Aut}(\mathcal{O}) = G_2(q)$ fixes the vector $\mathbf{1}$ for some prime power q ($G_2(q) < \Omega_7(q) < \Omega_8^+(q)$).

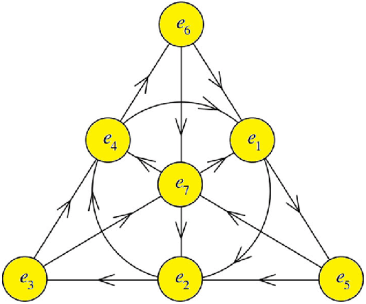


Figure: The multiplication table of $\mathcal{O}$ from the Fano plane

Octonions

Let N be the norm of $\mathcal{O}$ such that $N(x) = x\overline{x}$, where $\overline{x}$ is the octonion conjugate of x. Then N defines the quadric $Q^+(7, q)$, and the corresponding bilinear form B is induced from N:

$$B(x, y) = N(x + y) - N(x) - N(y), x, y \in \mathcal{O}.$$

The automorphism group $\mathrm{Aut}(\mathcal{O}) = G_2(q)$ fixes the vector $\mathbf{1}$ for some prime power q $(G_2(q) < \Omega_7(q) < \Omega_8^+(q))$.

The group $G_2(q)$ is the automorphism group of the classical generalized hexagon.

The symplectic basis for the octonions

Let $\{x_1, x_2, \ldots, x_8\}$ be the (symplectic) basis, which is more useful for looking at the subgroup structure. In particular, it works in the characteristic 2.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x_1					x_1	x_2	$-x_3$	x_4
x_2			$-x_1$	x_2			$-x_5$	x_6
x_3		x_1		x_3		$-x_5$		x_7
x_4	x_1			x_4		x_6	x_7	
x_5		x_2	x_3		x_5			x_8
x_6	$-x_2$		$-x_4$		x_6		x_8	
x_7	x_3	$-x_4$			x_7	$-x_8$		
x_8	$-x_5$	$-x_6$	x_7	x_8				

Table: The multiplication table of the new basis for $\mathcal{O}$

The octonion conjugate swaps x_4, x_5 and maps the other x_i's to their negation.

The minimal module of $F_4(q)$ (or $E_6(q)$)

Let $\mathcal{A}$ be a 27-dimensional algebra consisting of 3×3 Hermitian matrices over the octonions:

$$v = \begin{pmatrix} \lambda_0 & F & \overline{E} \\ \overline{F} & \lambda_0' & D \\ E & \overline{D} & \lambda_0'' \end{pmatrix} = (\lambda_0, \lambda_0', \lambda_0'' | D, E, F), \; D, E, F \in \mathcal{O}$$

where $\bar{\ }$ is the octonion conjugation. The multiplication is defined by $u \circ v = uv + vu$ for any $u, v \in \mathcal{A}$. The automorphism group of $\mathcal{A}$ is exactly the exceptional group $F_4(q)$.

Here the characteristic of $\mathbb{F}$ is not 2 or 3, $\mathcal{A}$ can be viewed as the Albert algebra (the exceptional Jordan algebra), Bray et al called $\mathcal{A}$ the Albert space.

The minimal module of $F_4(q)$ (or $E_6(q)$)

We take the following basis of $\mathcal{A}$:

$$w_0 = (1,0,0|0,0,0) \text{ and } w_i = (0,0,0|x_i,0,0),\ 0 < i \leq 8$$
$$w'_0 = (0,1,0|0,0,0) \text{ and } w'_i = (0,0,0|0,x_i,0),\ 0 < i \leq 8$$
$$w''_0 = (0,0,1|0,0,0) \text{ and } w''_i = (0,0,0|0,0,x_i),\ 0 < i \leq 8$$

Then we can write $v = \sum_{t=0}^{8}(\lambda_t w_t + \lambda'_t w'_t + \lambda''_t w''_t) \in \mathcal{A}$ with trace $\mathrm{Tr}(v) = \lambda_0 + \lambda'_0 + \lambda''_0$, and $I = w_0 + w'_0 + w''_0$.

Suppose q is odd. We derive a quadratic form from $\frac{1}{2}\mathrm{Tr}(v \circ v)$ as follows:

$$Q'(v) = \frac{1}{2}(\lambda_0^2 + \lambda_0'^2 + \lambda_0''^2) + \sum_{t=1}^{4}(\lambda_t\lambda_{9-t} + \lambda'_t\lambda'_{9-t} + \lambda''_t\lambda''_{9-t})$$

The minimal module of $F_4(q)$ (or $E_6(q)$)

Write $U = \{v \in \mathcal{A} : \mathrm{Tr}(v) = 0\}$, $U' = \langle I \rangle_{\mathbb{F}_q}$. Then Q_0 is an $F_4(q)$-invariant quadratic form on U induced by Q', which also works for even q:

$$Q_0(v) = \lambda_0^2 + \lambda_0\lambda_0' + \lambda_0'^2 + \sum_{t=1}^{4}(\lambda_t\lambda_{9-t} + \lambda_t'\lambda_{9-t}' + \lambda_t''\lambda_{9-t}'').$$

Define $W := U/(U \cap U')$, which is an irreducible minimal $F_4(q)$-module of dimension 25 or 26 according as the characteristic p is 3 or not. Let Q be the quadratic form induced from Q_0. Then

$$(W, Q) = \begin{cases} Q(24, q), \text{ if } p = 3; \\ Q^-(25, q), \text{ if } p = 2 \pmod 3); \\ Q^+(25, q), \text{ otherwise.} \end{cases}$$

The similarity group of the Dickson-Freudenthal determinant (a cubic form) is $\widetilde{E}_6(q)$, and $E_6(q) = \widetilde{E}_6(q)/Z(\widetilde{E}_6(q))$. The center has order 3 or 1 according as $q = 1 \pmod 3$ or not.

Theorem (Aschbacher, 1987)

There are three orbits of $\widetilde{E}_6(q)$ on nonzero vectors of $\mathcal{A}$, which are called the white, grey, black vectors, respectively.

Correspondingly, the 1-dimensional subspace spanned by a white (grey or black) vector is called a white (grey or black) point. The stabilizer of $I := w_0 + w_0' + w_0''$ in $\widetilde{E}_6(q)$ is $F_4(q)$.

M. Aschbacher. The 27-dimensional module for E_6. I. *Invent Math*, 89:159–195, 1987.

The characteristic $p = 3$

If $H \leq P\Gamma(V, \kappa)$ has exactly two orbits $\mathcal{M}_1, \mathcal{M}_2$ on the isotropic (singular) points of $\mathcal{P}_r$, $\mathcal{M}_1, \mathcal{M}_2$ are intriguing sets of the same type.

Corollary (Cohen and Cooperstein, 1988)

If $q = 3^f$, there are two $F_4(q)$-orbits on points of $W = Q(24, q)$. Since the two orbits have size

$$\frac{q^{12}-1}{q-1}(q^4+1) \text{ and } \frac{q^{12}-1}{q-1}(q^{12}-q^8)$$

respectively, which are not divisible by the ovoid number $q^{12}+1$. Hence they are tight sets.

A.M. Cohen and B.N. Cooperstein. The 2-spaces of the standard $E_6(q)$-module *Geometriae Dedicata*, 25:467–480, 1988.

The case $q = 2 \pmod 3$

Recall that all white points of $\mathcal{A}$ form an $\widetilde{E}_6(q)$-orbit. The group $F_4(q)$ split the orbit (white points) into two parts according as they are singular or nonsingular. Let M_1 be the white (singular) vectors in U, and denote $\mathcal{M}_1$ by the corresponding set of projective (white) points.

Theorem

If $q = 2 \pmod 3$, $\mathcal{M}_1$ is a $(q^4 + 1)$-tight set of $Q^-(25, q)$ with an automorphism group $F_4(q)$.

The case $q = 2 \pmod 3$

Lemma

There are $q^{15} + q^{12} - q^4 - 1$ elements of M_1 that are perpendicular to w_1.

Sketch of proof: the white vectors can be classified into the following three types:

(I) $v = f \cdot (A\overline{A}, B\overline{B}, 1 \mid \overline{B}, A, \overline{A}B)$ for some $f \in \mathbb{F}_q^*$ and $A, B \in \mathbb{O}$ such that $A\overline{A} + B\overline{B} + 1 = 0$,

(II) $v = e \cdot (C\overline{C}, 1, 0 \mid A, \overline{CA}, C)$ for some $e \in \mathbb{F}_q^*$ and octonions A, C such that $A\overline{A} = 0$, $C\overline{C} + 1 = 0$,

(III) $v = (0, 0, 0 \mid D, E, F)$, where D, E, F are octonions such that

$$D\overline{D} = E\overline{E} = F\overline{F} = 0, \quad DE = EF = FD = 0.$$

Then we enumerate the number of white vectors perpendicular to w_1.

Here we need to some facts about the left and right annihilators in $\mathcal{O}$.

Further direction

- When $q = 2$, there are exactly three orbits on W, all of which are tight sets.
- Question: In the case $q = 2 \pmod 3$, are other orbits tight sets? There exists one orbit with size $\frac{q^{12}-1}{q-1}(q^{12} - q^8)$. We conjecture that it is also a tight set.

Thanks for your attention!